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A Second-Order Delay Differential Equation with Multiple Periodic Solutions

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The equation $x''(t) + \omega^2 x(t) = bx([t-1])$, where $[\cdot]$ designates the greatest integer function, can be described in brief by two amazing properties. First, for certain values of the coefficients, some or all of its solutions are monotone although the corresponding homogeneous equation is clearly oscillatory. Second, for a specific relation between ω and b, there exist periodic solutions with different periods. © 1999 Academic Press

1. INTRODUCTION

The paper is concerned with the stability and oscillation properties of the equation

$$x''(t) + \omega^2 x(t) = bx([t-1]), \qquad b \neq 0, \tag{1}$$

whose coefficients are real constants and $[\cdot]$ signifies the greatest integer function. This investigation continues our earlier work on differential equations with piecewise constant arguments (EPCA), which was initiated in [1–6] and further developed by many authors in numerous publications. We refer the reader to [7] for a more comprehensive analysis of the subject and literature in this growing branch of functional differential equations. The success of this direction is motivated by the fact that EPCA represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations. Such hybrid systems are of considerable applied interest since they include, as particu-

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lar cases, impulsive and loaded equations of control theory and are similar to those found in some biomedical models. Although Eq. (1) fits within the general paradigm of delay differential or functional differential equations, the argument $[t-1]$ is a discontinuous function. Also note that the equation is nonautonomous, since the delay varies with t. In brief, this paper may be considered as a further attempt to extend the theory of functional differential equations with continuous arguments to differential equations with discontinuous arguments. The ideas of the paper have been induced by [2] and [6]. References to recent work on partial differential equations with piecewise continuous arguments may be found in [8] and [9].

DEFINITION 1. A solution of Eq. (1) on $[0, \infty)$ is a function $x(t)$ that satisfies the conditions:

(i) $x(t)$ is continuously differentiable on $[0, \infty)$;

(ii) $x''(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where it has one-sided limits;

(iii) Eq. (1) is satisfied on each interval $[n, n+1)$ with integral endpoints.

2. EXISTENCE OF SOLUTIONS

A typical EPCA, including Eq. (1), contains arguments that are constant on certain intervals. Continuity of a solution at a point joining any two consecutive intervals leads to recursion relations for the solution at such points. Hence, the solutions are determined by a finite set of initial data, rather than by an initial function as in the case of general functional differential equations. Therefore, underlying each equation with piecewise constant argument is a dynamical system governed by a difference equation of a discrete argument which describes its stability, oscillation, and periodic properties. It is not surprising then that recent work on EPCA has caused a new surge in the study of difference equations [5].

THEOREM 1. *Equation* (1) *has a solution on* $[0, \infty)$.

Proof. Denote by $x_n(t)$ the solution of Eq. (1) on the interval $n \le t < n + 1$ and let

$$x(n) = c_n, \qquad x(n-1) = c_{n-1}.$$

Then

$$x_n''(t) + \omega^2 x_n(t) = b c_{n-1},$$

whence

$$x_n(t) = A_n \cos \omega(t-n) + B_n \sin \omega(t-n) + \frac{b}{\omega^2} c_{n-1},$$

with arbitrary constants A_n and B_n. Putting $t = n$ here gives

$$c_n = A_n + \frac{b}{\omega^2} c_{n-1}, \qquad A_n = c_n - \frac{b}{\omega^2} c_{n-1},$$

and differentiating $x_n(t)$ at $t = n$ yields

$$B_n = \frac{d_n}{\omega}, \quad \text{where } d_n = x_n'(n).$$

Hence,

$$x_n(t) = \left(c_n - \frac{b}{\omega^2} c_{n-1}\right) \cos \omega(t-n) + \frac{1}{\omega} d_n \sin \omega(t-n) + \frac{b}{\omega^2} c_{n-1}, \tag{2}$$

$$x_n'(t) = -\omega\left(c_n - \frac{b}{\omega^2} c_{n-1}\right) \sin \omega(t-n) + d_n \cos \omega(t-n). \tag{3}$$

At $t = n + 1$ it follows from (2) and (3) that

$$c_{n+1} = c_n \cos \omega + \frac{1}{\omega} d_n \sin \omega + \frac{b}{\omega^2}(1 - \cos \omega) c_{n-1}, \tag{4}$$

$$d_{n+1} = -\omega c_n \sin \omega + d_n \cos \omega + \frac{b}{\omega} c_{n-1} \sin \omega. \tag{5}$$

Now we introduce the vector $v_n = col(c_n, d_n)$ and the matrices

$$A = \begin{pmatrix} \cos \omega & \sin \omega/\omega \\ -\omega \sin \omega & \cos \omega \end{pmatrix}, \qquad B = \begin{pmatrix} b(1-\cos \omega)/\omega^2 & 0 \\ b \sin \omega/\omega & 0 \end{pmatrix},$$

and write

$$v_{n+1} = A v_n + B v_{n-1}. \tag{6}$$

We look for a nonzero solution of this difference equation in the form $v_n = \lambda^n k$, with a constant vector k, and conclude that λ satisfies the equation

$$\det(\lambda^2 I - \lambda A - B) = 0, \tag{7}$$

which is changed to

$$\begin{vmatrix} \lambda^2 - \lambda\cos\omega - b(1-\cos\omega)\omega^{-2} & -\lambda\sin\omega/\omega \\ \lambda\omega\sin\omega - b\sin\omega/\omega & \lambda^2 - \lambda\cos\omega \end{vmatrix} = 0.$$

Calculations lead to the equation

$$\lambda^3 - 2\lambda^2\cos\omega + \left(1 - \frac{(1-\cos\omega)b}{\omega^2}\right)\lambda - \frac{b(1-\cos\omega)}{\omega^2} = 0 \quad (8)$$

that has three nontrivial solutions if

$$1 - \cos\omega \neq 0. \quad (9)$$

Assuming that these roots are simple, we write the general solution of Eq. (6),

$$v_n = \lambda_1^n k_1 + \lambda_2^n k_2 + \lambda_3^n k_3, \quad (10)$$

with constant vectors k_j each of which depends on the corresponding value λ_j and contains one arbitrary scalar factor. These factors can be found from adequate initial or boundary conditions. If some λ_j is a multiple zero of Eq. (8), then the expression for v_n also includes products of λ^n by n or n^2. Finally, the solution $x_n(t)$ is obtained by substituting the appropriate components of the vectors v_n and v_{n-1} in (2). ∎

Remark 1. Clearly, we can eliminate d_n and d_{n+1} from Eqs. (4) and (5) and derive the equation

$$c_{n+2} - 2c_{n+1}\cos\omega + \left(1 - \frac{(1-\cos\omega)b}{\omega^2}\right)c_n - \frac{b(1-\cos\omega)}{\omega^2}c_{n-1} = 0. \quad (11)$$

THEOREM 2. *The three-point boundary value problem*

$$x(-1) = c_{-1}, \qquad x(0) = c_0, \qquad x(N-1) = c_{N-1} \quad (12)$$

for Eq. (1) *has a unique solution on* $0 \leq t < \infty$ *if* $N > 1$ *is an integer and the following hypotheses are satisfied.*

(i) *The characteristic roots* λ_j *are nontrivial and distinct.*

(ii) $(\lambda_2^N - \lambda_1^N)/(\lambda_2 - \lambda_1) \neq (\lambda_3^N - \lambda_1^N)/(\lambda_3 - \lambda_1)$.

(iii) $\cos\omega \neq 1$.

Proof. Formula (10) furnishes for the components c_n of the vectors v_n the representation

$$c_n = \lambda_1^n k_{11} + \lambda_2^n k_{21} + \lambda_3^n k_{31}, \tag{13}$$

with arbitrary constants k_{ij}. If the values c_{-1}, c_0, and c_{N-1} are given, the coefficients k_{ij} satisfy the system of equations

$$\begin{gathered} \lambda_1^{-1} k_{11} + \lambda_2^{-1} k_{21} + \lambda_3^{-1} k_{31} = c_{-1}, \qquad k_{11} + k_{21} + k_{31} = c_0, \\ \lambda_1^{N-1} k_{11} + \lambda_2^{N-1} k_{21} + \lambda_3^{N-1} k_{31} = c_{N-1}, \end{gathered} \tag{14}$$

the determinant of which is different from zero, by virtue of hypothesis (ii). Hence, we can find the coefficients k_{ij} and the components c_n uniquely. Condition (iii) merely restates inequality (9) which ensures that the roots λ_j are not zero. Furthermore, once the values c_n have been found, we calculate the components d_n from Eq. (4) and then substitute both c_n and d_n in Eq. (2). For $N = 2$, hypothesis (ii) is part of (i). ∎

THEOREM 3. *If the characteristic roots λ_j are nontrivial, $\lambda_1 = \lambda_3$, and*

$$\left(\lambda_2^N - \lambda_1^N\right)/\left(\lambda_2 - \lambda_1\right) \neq N\lambda_1^{N-1}, \tag{15}$$

then the boundary value problem (12) *for Eq.* (1) *has a unique solution on* $[0, \infty)$.

THEOREM 4. *If $\lambda_1 = \lambda_2 = \lambda_3$, then problem* (12) *for Eq.* (1) *has a unique solution on* $[0, \infty)$.

Remark 2. If $\omega = 2\pi j$, $j \neq 0$, the characteristic equation (8) has only two nonzero roots $\lambda_1 = \lambda_2 = 1$, and in this case a two-point boundary value problem is posed for Eq. (1).

THEOREM 5. *If $\omega = 2\pi j$, where $j \neq 0$ is an integer, then the problem*

$$x(-1) = c_{-1}, \qquad x(0) = c_0, \tag{16}$$

for Eq. (1) *has infinitely many solutions on* $[0, \infty)$, *and each solution is a periodic function with period* 1 *for* $1 \leq t < \infty$.

Proof. Equation (1) on the interval $0 \leq t < 1$ becomes

$$x_0''(t) + \omega^2 x_0(t) = bc_{-1},$$

whence

$$x_0(t) = \left(c_0 - \frac{b}{\omega^2} c_{-1}\right) \cos \omega t + \frac{1}{\omega} d_0 \sin \omega t + \frac{b}{\omega^2} c_{-1},$$

with the notation

$$d_0 = x_0'(0).$$

For $1 \le t < 2$, we have the equation

$$x_1''(t) + \omega^2 x_1(t) = bc_0$$

and find the solution

$$x_1(t) = \left(c_0 - \frac{b}{\omega^2}c_0\right)\cos \omega(t-1) + \frac{1}{\omega}d_0 \sin \omega(t-1) + \frac{b}{\omega^2}c_0$$

satisfying the conditions

$$x_1(1) = x_0(1) = c_0, \qquad x_1'(1) = x_0'(1) = d_0.$$

In general, for $\cos \omega = 1$ and $n \ge 0$, it follows from Eqs. (4) and (5) that

$$c_{n+1} = c_n, \qquad d_{n+1} = d_n,$$

and so,

$$c_n = c_0, \qquad d_n = d_0, \qquad n \ge 0.$$

Substituting these values in Eq. (2) yields the solution

$$x_n(t) = \left(c_0 - \frac{b}{\omega^2}c_0\right)\cos \omega(t-n) + \frac{1}{\omega}d_0 \sin \omega(t-n) + \frac{b}{\omega^2}c_0 \tag{17}$$

on the interval $n \le t < n+1$, with $n \ge 1$. This formula shows that the solution includes an arbitrary constant d_0. ∎

Remark 3. For large ω, the term containing d_0 is small and the dominant terms in the solution formula (17) depend only on c_0.

THEOREM 6. *The boundary value problem*

$$x(-1) = c_{-1}, \qquad x(0) = c_0, \qquad x'(0) = d_0, \tag{18}$$

for Eq. (1) *has a unique solution on* $[0,\infty)$.

3. OSCILLATIONS AND STABILITY: SPECIAL CASE $\omega = 0$

Letting $\omega \to 0$ in Eq. (8) yields the characteristic equation

$$\lambda^3 - 2\lambda^2 + \left(1 - \frac{b}{2}\right)\lambda - \frac{b}{2} = 0 \tag{19}$$

for Eq. (1), with $\omega = 0$. It is interesting to consider problem (1), (12) in this case. Note that formula (2) for the solution of Eq. (1) was derived with the implicit assumption $\omega \neq 0$. Writing Eq. (2) as

$$x_n(t) = c_n \cos \omega(t-n) + \frac{b}{\omega^2}(1 - \cos \omega(t-n))c_n + \frac{1}{\omega} d_n \sin \omega(t-n)$$

and letting $\omega \to 0$ yields the solution

$$x_n(t) = \frac{b}{2} c_{n-1}(t-n)^2 + d_n(t-n) + c_n \tag{20}$$

of Eq. (1) for $\omega = 0$.

THEOREM 7. *If $\omega = 0$, $b < 0$, then every solution of Eq.* (1) *oscillates in* $[0, \infty)$ *and is either unbounded or tends to zero as $t \to +\infty$.*

Proof. It has been proved in [7] that all solutions of a linear system with piecewise constant arguments and constant coefficients oscillate if and only if the corresponding characteristic equation has no positive roots. This is true for Eq. (19) when $b < 0$ since it can be written as

$$\lambda(\lambda - 1)^2 - \frac{b}{2}(\lambda + 1) = 0. \tag{21}$$

The inequalities $P(-1) < 0$ and $P(0) > 0$, where $P(\lambda)$ designates the left-hand side of Eq. (19), show that this equation has a root $\lambda_1 \in (-1, 0)$. Furthermore, the Descartes rule of signs confirms that λ_1 is the only real root of Eq. (19). Next, from the equation $\lambda_1 + \lambda_2 + \lambda_3 = 2$ we conclude that $\lambda_2 + \lambda_3 > 2$, $\operatorname{Re} \lambda_2 = \operatorname{Re} \lambda_3 > 1$, and

$$|\lambda_2| = |\lambda_3| > 1.$$

With regard to Eq. (13), it means that $c_n \to \infty$ as $n \to \infty$, provided the boundary values c_{-1}, c_0, c_{N-1} are chosen so that in the solution of system (14) the values of k_{21} and k_{31} are not zero simultaneously. On the other hand, taking any $c_0 \neq 0$ and selecting $c_{-1} = \lambda_1^{-1} c_0$, $c_{N-1} = \lambda_1^{N-1} c_0$ gives $k_{11} = c_0$, $k_{21} = k_{31} = 0$. Since $|\lambda_1| < 1$, we have in this case $x_n(t) \to 0$ as $t \to \infty$. ∎

THEOREM 8. *For $\omega = 0$ and*

$$0 < b < (71 - 17^{3/2})/4, \tag{22}$$

each solution of Eq. (1) *is nonoscillatory in* $[0, \infty)$ *and is either unbounded or tends to zero as $t \to +\infty$.*

Proof. Solving Eq. (21) with respect to the parameter b produces the function

$$b(\lambda) = 2\lambda(\lambda - 1)^2/(\lambda + 1), \tag{23}$$

with the derivative

$$b'(\lambda) = 2(\lambda - 1)(2\lambda^2 + 3\lambda - 1)/(\lambda + 1)^2,$$

whose zeros

$$\lambda_1' = (-3 - 17^{1/2})/4, \qquad \lambda_2' = 1, \qquad \lambda_3' = (-3 + 17^{1/2})/4$$

yield two local minima and a maximum, respectively, of $b(\lambda)$. Calculations show that

$$b(\lambda_3') = (71 - 17^{3/2})/4 \approx 0.2268.$$

Each line $b = const.$ in the upper half-plane (λ, b) satisfying inequalities (22) intersects the curve (23) at three points with two abscissas in the interval $(0, 1)$ and one in $\lambda > 1$. Hence, under conditions (22), Eq. (19) has three positive roots $\lambda_1, \lambda_2, \lambda_3$ such that $0 < \lambda_1$, $\lambda_2 < 1$, $\lambda_3 > 1$, which proves that the variable $c_n = x_n(n)$ retains its sign, for large n.

Hence, the solution (parabola) $x_n(t)$ given by Eq. (20) is nonoscillatory for large n if it does not intersect the interval $n < t < n + 1$ twice. Assuming the opposite, we must conclude that the derivatives $d_n = x_n'(n)$ and $d_{n+1} = x_n'(n+1)$ have different signs. On the other hand, the expression

$$x_n'(t) = bc_{n-1}(t - n) + d_n$$

at $t = n + 1$ gives

$$d_{n+1} - d_n = bc_{n-1},$$

whence

$$d_{n+1} = d_0 + \sum_{i=0}^{n} bc_{i-1}.$$

Since the sum on the right becomes monotone, starting with some n, it has a limit (finite or infinite) which implies that d_n preserves its sign, for large n. This proves that all solutions of Eq. (1) are nonoscillatory.

Depending on the boundary conditions, it may happen that $k_{31} \neq 0$, and in this case the corresponding solutions of Eq. (1) are unbounded. On the contrary, the case $k_{31} = 0$ generates solutions that go to zero. ∎

THEOREM 9. *For $\omega = 0$ and*

$$(71 - 17^{3/2})/4 < b < 6, \tag{24}$$

each solution of Eq. (1) *is either unbounded nonoscillatory, or oscillatory and approaching zero as $t \to +\infty$.*

Proof. For the values of b satisfying conditions (24), Eq. (19) has only one real root $\lambda_1 > 1$ and two complex roots λ_2, λ_3. Let

$$|\lambda_2| = |\lambda_3| = r.$$

From the equation $\lambda_1 \lambda_2 \lambda_3 = b/2$ we have $r^2 = b/2\lambda_1$, and therefore consider the function

$$r^2(\lambda) = (\lambda - 1)^2/(\lambda + 1), \qquad \lambda > 0, \tag{25}$$

generated by Eq. (23). Over the interval $0 < \lambda < 3$, the graph of $r^2(\lambda)$ in the plane (λ, r^2) lies below the line $r^2 = 1$, with values $r^2(0^+) = 1$ and $r^2(3) = 1$ at the endpoints. For $\lambda > 3$, the graph grows to infinity as $\lambda \to \infty$ approaching the slant asymptote $r^2 = \lambda - 3$. The line $b = b_0 = (71 - 17^{3/2})/4$ in the plane (λ, b) meets the graph of $b(\lambda)$ at two points with abscissas $0 < \lambda_3' < 1$ and $1 < \lambda_0 < 2$. In the domain $\lambda > \lambda_0$, $b > b_0$, the function $b(\lambda)$ has an increasing inverse $\lambda(b)$ and, furthermore, $r(\lambda) < 1$ for $\lambda_0 < \lambda < 3$. Also, from Eq. (23) we see that $b(3) = 6$. Therefore, λ increases from λ_0 to 3 as b runs through the interval (24), and in this case $r < 1$. Under appropriate boundary conditions, the characteristic root $\lambda_1 > 1$ implies the existence of monotone unbounded solutions to Eq. (1), whereas the complex roots λ_2 and λ_3 lead to oscillatory solutions. The latter tend to zero since λ_2 and λ_3 lie in the unit disk $|\lambda| < 1$. ∎

THEOREM 10. *For $\omega = 0$ and $b = 6$, each solution of Eq.* (1) *is either unbounded nonoscillatory or periodic with period* 3. *The number* 6 *is the only value of the parameter b for which the equation*

$$x''(t) = bx([t - 1]) \tag{26}$$

has periodic solutions.

Proof. In the given case, Eq. (19) becomes

$$\lambda^3 - 2\lambda^2 - 2\lambda - 3 = 0,$$

which can be written as

$$(\lambda - 3)(\lambda^2 + \lambda + 1) = 0.$$

The root $\lambda_1 = 3$ is a source of unbounded nonoscillatory solution for Eq. (1) while the roots λ_2 and λ_3 satisfy the relations $\lambda_1^3 = \lambda_2^3 = 1$ and generate, under appropriate boundary conditions, solutions of period 3 for Eq. (1). Conversely, assume that Eq. (1) has a periodic solution. Since $\lambda = 1$ and $\lambda = -1$ do not satisfy Eq. (19), there exist two complex roots λ_2 and λ_3 such that $|\lambda_2| = |\lambda_3| = 1$. From the equality $\lambda_1\lambda_2\lambda_3 = b/2$ it follows that $|\lambda_1| = |b|/2$. The value $\lambda = -b/2$ does not satisfy Eq. (19), and so $\lambda_1 = b/2$. Substituting this number in Eq. (19) gives the identity

$$\frac{b^3}{8} - \frac{b^2}{2} + \frac{b}{2} - \frac{b^2}{4} - \frac{b}{2} = 0,$$

whence $b = 6$. ∎

THEOREM 11. *For $b > 6$, the solutions of Eq.* (26) *are unbounded. Depending on the boundary conditions* (12), *they are either oscillatory or nonoscillatory.*

Proof. For $b > 6$, Eq. (19) has a positive root $\lambda_1 > 3$ and two roots λ_2, λ_3 which are either negative or complex. If λ_2 and λ_3 are complex, then from Eq. (25) it follows that $r^2(\lambda) > 1$, which implies unboundedness of the solutions to Eq. (26). We have noted in the proof of Theorem 8 that the function $b(\lambda)$ defined by Eq. (23) attains local minimum at $\lambda_1' = (-3 - 17^{1/2})/4$. Calculations show that

$$b(\lambda_1') = (71 + 17^{3/2})/4 \approx 35.2732.$$

Therefore, λ_2 and λ_3 are complex for $6 < b < b(\lambda_1')$. On the other hand, λ_2 and λ_3 are negative if $b \geq b(\lambda_1')$, and in this case $\lambda_2 < -1$, $\lambda_3 < -1$, which shows that all solutions of (26) are unbounded. Furthermore, the positive root is a source of nonoscillatory solutions, and the negative or complex roots generate oscillatory solutions. ∎

4. GENERAL CASE: $\omega \neq 0$

In this section, we modify the technique of the preceding part since the characteristic equation (8) contains two parameters, b and ω. It is also natural to expect that the solutions behavior becomes more complicated, which is indeed true.

THEOREM 12. *For $b < 0$, all solutions of Eq.* (1) *oscillate in* $[0, \infty)$.

Proof. The rule of signs confirms that the characteristic polynomial

$$P(\lambda) = \lambda^3 - 2\lambda^2 \cos\omega + \left(1 - \frac{b(1 - \cos\omega)}{\omega^2}\right)\lambda - \frac{b(1 - \cos\omega)}{\omega^2} \tag{27}$$

has no positive roots when $\cos\omega < 0$, and therefore all solutions of Eq. (1) oscillate. This conclusion remains valid if $\cos\omega > 0$ since the parabola

$$f(\lambda) = \lambda^2 - 2\lambda\cos\omega + 1 - \frac{b(1 - \cos\omega)}{\omega^2},$$

attaining the positive minimum

$$f(\cos\omega) = (1 - \cos\omega)\left(1 + \cos\omega - \frac{b}{\omega^2}\right),$$

intersects the hyperbola

$$g(\lambda) = \frac{b(1 - \cos\omega)}{\lambda\omega^2}$$

at a single point, with a negative abscissa. ∎

THEOREM 13. *For $b < 0$, all solutions of Eq* (1) *tend to zero as $t \to +\infty$ if and only if*

$$\cos\omega < -\frac{1}{2} \quad \textit{and} \quad \frac{b}{\omega^2} > \frac{1 + 2\cos\omega}{1 - \cos\omega}. \tag{28}$$

Proof. For $b < 0$, the polynomial (27) has a zero $\lambda_1 \in (-1, 0)$ since

$$P(0) = -\frac{b(1 - \cos\omega)}{\omega^2} > 0,$$

$$P(-1) = -2 + 2\cos\omega < 0.$$

On the other hand, we have

$$P(-1-\varepsilon) = -(1+\varepsilon)^3 - 2(1+\varepsilon)^2 \cos\omega - (1+\varepsilon) + \frac{b\varepsilon(1-\cos\omega)}{\omega^2}$$
$$< -(1+\varepsilon)^3 + 2(1+\varepsilon)^2 - (1+\varepsilon)$$
$$= -\varepsilon^2(1+\varepsilon) < 0,$$

for any $\varepsilon > 0$, which implies that $P(\lambda)$ has no zero $\lambda_i < -1$. Further, the zeros λ_i satisfy the relation

$$\lambda_1\lambda_2\lambda_3 = \frac{b(1-\cos\omega)}{\omega^2},$$

and from Eq. (8) we find

$$\frac{b(1-\cos\omega)}{\omega^2} = \frac{\lambda(\lambda^2 - 2\lambda\cos\omega + 1)}{\lambda+1}. \tag{29}$$

Hence,

$$\lambda_2\lambda_3 = \frac{\lambda_1^2 - 2\lambda_1\cos\omega + 1}{\lambda_1 + 1}. \tag{30}$$

The condition $|\lambda_i| < 1$ $(i = 1, 2, 3)$ is necessary and sufficient for all solutions of Eq. (1) to go to zero as $t \to +\infty$, which is equivalent to the inequality

$$\left(\lambda_1^2 - 2\lambda_1\cos\omega + 1\right)/(\lambda_1 + 1) < 1. \tag{31}$$

From here,

$$\lambda_1 > 1 + 2\cos\omega,$$

and now we use the inequality

$$b(1-\cos\omega)/\lambda_1\omega^2 < 1,$$

in order to obtain

$$b/\omega^2 > \lambda_1/(1-\cos\omega) > (1+2\cos\omega)/(1-\cos\omega).$$

∎

THEOREM 14. *All solutions of Eq.* (1) *are unbounded as $t \to +\infty$ if*

$$\frac{b(1-\cos\omega)}{\omega^2} > 1 \quad \text{and} \quad \cos\omega < 0. \tag{32}$$

Proof. Inequalities (32) imply the existence of a single positive root λ_1 of the polynomial (27). In addition, since $P(0) < 0$ and

$$P(1) = 2(1 - \cos\omega)\left(1 - \frac{b}{\omega^2}\right) < 0,$$

then $\lambda_1 > 1$. The inequality $\lambda_2 \lambda_3 \le 1$ is impossible since in this case Eq. (30) gives $\lambda_1 \le 1 + 2\cos\omega$ which contradicts $\lambda_1 > 1$. In turn, the inequality $\lambda_2 \lambda_3 > 1$ implies that $|\lambda_2| > 1$ and $|\lambda_3| > 1$ if λ_2 and λ_3 are complex. Finally, if λ_2 and λ_3 are negative, we must conclude that $\lambda_2 < -1$ and $\lambda_3 < -1$. Indeed, from the inequalities $P(0) < 0$ and $P(-1) = -2 - 2\cos\omega < 0$, it follows that either $-1 < \lambda_2 < 0$ and $-1 < \lambda_3 < 0$ or $\lambda_2 < -1$ and $\lambda_3 < -1$. However, the first case should be dismissed since $\lambda_2 \lambda_3 > 1$. The inequalities $|\lambda_i| > 1$ $(i = 1, 2, 3)$ confirm that each solution of Eq. (1) is unbounded as $t \to +\infty$. ∎

THEOREM 15. *For $b > 0$, all solutions of Eq.* (1) *tend to zero as $t \to +\infty$ if and only if either*

$$\cos\omega > 0 \quad \textit{and} \quad b < \omega^2 \tag{33}$$

or

$$-\frac{1}{2} < \cos\omega < 0 \quad \textit{and} \quad \frac{b}{\omega^2} < \frac{1 + 2\cos\omega}{1 - \cos\omega}. \tag{34}$$

Proof. By virtue of the inequality $\cos\omega > 0$, the polynomial $P(\lambda)$ has either one or three positive roots. Since $P(0) < 0$ and

$$P(1) = 2(1 - \cos\omega)\left(1 - \frac{b}{\omega^2}\right),$$

the condition $b < \omega^2$ implies the existence of a positive root $\lambda_1 < 1$. Conditions (33) also show that $P(\lambda)$ has no negative roots. From $P(0) < 0$ and $P(1) > 0$, it follows that $P(\lambda)$ has either one or three zeros in $(0, 1)$, and if the latter case holds true, the first part of the theorem is proved. On the other hand, the equation $\lambda_1 + \lambda_2 + \lambda_3 = 2\cos\omega$ indicates that the inequalities $\lambda_2 > 1$ and $\lambda_3 > 1$ cannot occur simultaneously, and therefore it remains to consider the case when λ_2 and λ_3 are complex. From Eq. (30) we conclude that the inequality $|\lambda_2 \lambda_3| < 1$ takes place for $\lambda_1 < 1 + 2\cos\omega$; hence, it is also valid for $\lambda_1 < 1$.

For $\cos\omega < 0$ and $0 < b/\omega^2 < 1$, the polynomial $P(\lambda)$ has a single positive zero λ_1, and it lies in $(0, 1)$. Next, we have

$$P(-1 + \varepsilon) = -(1 - \varepsilon)\left[(1 - \varepsilon)^2 + 2(1 - \varepsilon)\cos\omega + 1\right] - \frac{b\varepsilon(1 - \cos\omega)}{\omega^2}$$

and since $(1-\varepsilon)^2 + 2(1-\varepsilon)\cos\omega + 1 > 0$, then $P(-1+\varepsilon) < 0$, for $0 < \varepsilon < 1$. In other words, $P(\lambda)$ has no zero in $(-1, 0)$. The case $\lambda_2 < -1$ and $\lambda_3 < -1$ is impossible since the inequality $\lambda_2 \lambda_3 > 1$ implies

$$\left(\lambda_1^2 - 2\lambda_1 \cos\omega + 1\right)/(\lambda_1 + 1) > 1,$$

that is, $\lambda_1 > 1 + 2\cos\omega$. Hence, it follows from $b(1-\cos\omega)/\lambda_1 \omega^2 > 1$ that

$$b/\omega^2 > (1 + 2\cos\omega)/(1-\cos\omega),$$

which contradicts inequality (34). Finally, for complex λ_2 and λ_3, the assumption $|\lambda_2 \lambda_3| \geq 1$ leads to $\lambda_1 \geq 1 + 2\cos\omega$. On the other hand, from the inequality

$$|\lambda_2 \lambda_3| = \frac{b(1-\cos\omega)}{\lambda_1 \omega^2} \geq 1,$$

it follows that

$$b(1-\cos\omega)/\omega^2 \geq \lambda_1 \geq 1 + 2\cos\omega,$$

which again contradicts inequality (34). ∎

Let us discuss the oscillation properties of Eq. (1) when $b > 0$. In this case, the polynomial $P(\lambda)$ given by Eq. (27) may have either one or three positive roots. The inequalities $b > 0$, $\cos\omega < 0$ guarantee the existence of a single positive root which implies that some solutions of Eq. (1) (generated by the negative or complex roots of $P(\lambda)$) oscillate. It is therefore interesting to find out whether there exist values of $b > 0$ and $\cos\omega > 0$ such that *all* solutions of Eq. (1) are nonoscillatory.

THEOREM 16. *Let* $0 < \omega < \pi/2$, $4/5 < \cos\omega < 1$, *and* $0 < b < \omega^2$. *Then each solution of Eq.* (1) *is nonoscillatory.*

Proof. Subject to hypotheses (33), the polynomial $P(\lambda)$ given by Eq. (27) has a positive root $\lambda_1 < 1$ and no negative roots. If $P(\lambda)$ has three positive roots, all of them lie in $(0, 1)$, and in this case the derivative $P'(\lambda)$ has two zeros in the same interval. The roots of the equation

$$P'(\lambda) = 3\lambda^2 - 4\lambda\cos\omega + \left(1 - \frac{b(1-\cos\omega)}{\omega^2}\right) = 0$$

are real distinct if

$$\frac{b(1-\cos\omega)}{\omega^2} > 1 - \frac{4}{3}\cos^2\omega,$$

and since $b < \omega^2$, then $\cos \omega$ necessarily satisfies the inequality

$$1 - \frac{4}{3}\cos^2 \omega < 1 - \cos \omega,$$

whence $\cos \omega > 3/4$. Furthermore, the zeros of $P'(\lambda)$ lie in $(0,1)$ if $b < 4\omega^2$, a condition weaker than $b < \omega^2$.

From Eq. (8) it follows that

$$B(\lambda) = \frac{\lambda(\lambda^2 - 2\lambda\cos\omega + 1)}{\lambda + 1}, \tag{35}$$

where

$$B(\lambda) = b(1 - \cos\omega)/\omega^2,$$

and we explore the curve (35) in the (λ, B) plane. The derivative of $B(\lambda)$ is

$$B'(\lambda) = Q(\lambda)/(\lambda + 1)^2,$$

with

$$Q(\lambda) = 2\lambda^3 + (3 - 2\cos\omega)\lambda^2 - 4\lambda\cos\omega + 1. \tag{36}$$

The derivative

$$Q'(\lambda) = 6\lambda^2 + 2(3 - 2\cos\omega)\lambda - 4\cos\omega$$

shows that $Q(\lambda)$ has a maximum at $\lambda = -1$ and a minimum at $\lambda = (2\cos\omega)/3$, with values

$$Q(-1) = 2(1 + \cos\omega),$$
$$Q\left(\tfrac{2}{3}\cos\omega\right) = -\tfrac{8}{27}\cos^3\omega - \tfrac{4}{3}\cos^2\omega + 1.$$

If $Q(2\cos\omega/3) < 0$, i.e.,

$$\tfrac{8}{27}\cos^3\omega + \tfrac{4}{3}\cos^2\omega - 1 > 0, \tag{37}$$

then the graph of $Q(\lambda)$ intersects the interval $(0,1)$ twice since $Q(0) = 1 > 0$ and $Q(1) = 6(1 - \cos\omega) > 0$. This implies that $B(\lambda)$ has a maximum $B_M = B(\lambda_M)$ and a minimum $B_m = B(\lambda_m)$, where $0 < \lambda_M < \lambda_m < 1$. Since $B(1) = 1 - \cos\omega$, which is equivalent to $b = \omega^2$, it follows from the condition $b < \omega^2$ that any horizontal line $B(\lambda) = C$ such that $B_m < C < B_M$ crosses the graph of $B(\lambda)$ at three points, with abscissas in $(0,1)$. This means that Eq. (8) has three roots in the interval $(0,1)$. The substitution $u = (2\cos\omega)/3$ changes inequality (37) to

$$u^3 + 3u^2 - 1 > 0. \tag{38}$$

Clearly, Eq. (38) has a single positive root u_0, and therefore inequality (38) holds true for $u > u_0$. We know already that $\cos\omega > 3/4$, and so $u_0 > 1/2$. At $u_1 = 1/2$ and $u_2 = 7/12$, we have $1/8 + 3/4 - 1 < 0$ and $343/1728 + 49/48 - 1 > 0$. In fact, $u_0 = 0.5321$ and we conclude that inequality (37) is valid for $0.8 < \cos\omega < 1$. Since all roots of $P(\lambda)$ are positive, the variable $x(n) = c_n$ retains its sign, for large n. We want to show that for such n, the integral curve (2) does not intersect the interval $[n, n+1]$. Assuming the opposite implies that $x_n(t)$ crosses $[n, n+1]$ an even number of times since $c_n \cdot c_{n+1} > 0$. Hence, there exist points $t_1, t_2 \in [n, n+1]$ such that $x_n(t_1) = 0$ and $x_n(t_2) = 0$. Keeping in mind that $0 < t_i - n < 1$ $(i = 1, 2)$ and $0 < \omega < \pi/2$, we have $0 < \omega(t_i - n) < \pi/2$, and turning to Eq. (2) see that the equation

$$\left(c_n - \frac{b}{\omega^2}c_{n-1}\right)\cos\theta + \frac{1}{\omega}d_n\sin\theta + \frac{b}{\omega^2}c_{n-1} = 0$$

must have at least two solutions, $\theta_i = \omega(t_i - n)$, in $(0, \pi/2)$. This is impossible, which proves that each solution of Eq. (1) is nonoscillatory. ∎

COROLLARY 1. *With the hypotheses of Theorem* 16, *each solution of Eq*. (1) *tends to zero monotonically as* $t \to +\infty$.

THEOREM 17. *Let* $0 < \omega < \pi/2$ *and*

$$1 < \frac{b}{\omega^2} < \frac{1}{1 - \cos\omega}. \tag{39}$$

Then each solution of Eq. (1) *is either eventually monotone unbounded or oscillating and approaching zero*.

COROLLARY 2. *Assuming that* $0 < \omega < \pi/2$, *each solution of Eq*. (1) *is oscillatory if and only if* $b < 0$.

Finally, we discuss the existence of periodic solutions to Eq. (1). Note that the functions

$$\cos\omega(t - n) = \cos\omega(t - [t]) \quad \text{and} \quad \sin\omega(t - n) = \sin\omega(t - [t])$$

are periodic with period 1 since $0 \le t - [t] < 1$. Further, the coefficients c_n and d_n in the solution formula (2) are the components of vectors (10) which are represented as linear combinations of the powers λ_i^n of the characteristic roots λ_i. Since the coefficients in Eq. (10) depend only on the boundary conditions (12), Eq. (1) has a periodic solution if and only if there exists an eigenvalue λ_j which is a root of unity. Let us rewrite Eq. (8) in the form

$$\lambda^3 - 2\lambda^2\cos\omega + (1 - B)\lambda - B = 0, \tag{40}$$

where $B = b(1 - \cos \omega)/\omega^2$, and assume that (40) has two complex zeros λ_2 and λ_3 which are roots of unity. From the equation $\lambda_1 \lambda_2 \lambda_3 = B$ for the roots of Eq. (40), it follows that $\lambda_1 = B$ since $\lambda_2 \lambda_3 = 1$. In other words, the parameter B is also a root of Eq. (40), i.e.,

$$B^3 - 2B^2 \cos \omega + (1 - B)B - B = 0,$$

or

$$B^3 - (1 + 2\cos \omega)B^2 = 0.$$

Hence,

$$B = 1 + 2\cos \omega \tag{41}$$

and

$$b = \frac{(1 + 2\cos \omega)\omega^2}{1 - \cos \omega}. \tag{42}$$

In this case, Eq. (40) becomes

$$(\lambda - B)(\lambda^2 + \lambda + 1) = 0, \tag{43}$$

and since the zeros of the second factor in Eq. (43) are the complex roots of the equation $\lambda^3 = 1$, we arrive at the following conclusion.

THEOREM 18. *Condition* (42) *is necessary and sufficient for the existence of periodic solutions with period* 3 *to Eq*. (1).

Note that the condition $b = 6$ for the existence of periodic solutions with period 3 to Eq. (26) follows from Eq. (42) as $\omega \to 0$. Furthermore, the only real eigenvalues that generate periodic solutions of Eq. (1) are $\lambda = 1$ or $\lambda = -1$. If $B = 1$, then $\cos \omega = 0$ and

$$\omega = (2j - 1)\pi/2, \qquad b = \omega^2, \quad j = \pm 1, \pm 2, \dots . \tag{44}$$

If $B = -1$, then $\cos \omega = -1$ and

$$\omega = (2j - 1)\pi, \qquad b = -\omega^2/2. \tag{45}$$

THEOREM 19. *If hypotheses* (44) *hold true, each solution of Eq*. (1) *is periodic and is either constant or has period* 3.

THEOREM 20. *If hypotheses* (45) *hold true, each solution of Eq*. (1) *is periodic with period* 6. *There also exist solutions with periods* 2 *or* 3.

THEOREM 21. *The condition* $b = \omega^2$ *is necessary and sufficient for the existence of constant solutions to Eq*. (1).

THEOREM 22. *The condition* $\omega = (2j - 1)\pi$ *is necessary and sufficient for the existence of periodic solutions with period* 2 *to Eq*. (1).

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